Diagonalization

Theorem 8

Let A be an $n \times n$ matrix with eigenvalues λ_1, λ_2, λ_3,… λ_n. If A has n independent eigenvectors $\vec{v}^{(1)}$, $\vec{v}^{(2)}$, $\vec{v}^{(3)}$, … , $\vec{v}^{(n)}$, then there exists a matrix P such that

$$P^{-1}AP = D \qquad \textbf{(8)}$$

where

$$P = \begin{bmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} & \vec{v}^{(3)} & \dots & \vec{v}^{(n)} \end{bmatrix}_{n\times n} \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

Proof

$$P\,D = \begin{bmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots & \vec{v}^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\vec{v}^{(1)} & \lambda_2\vec{v}^{(2)} & \cdots & \lambda_n\vec{v}^{(n)} \end{bmatrix}$$

$$= \begin{bmatrix} A\vec{v}^{(1)} & A\vec{v}^{(2)} & \cdots & A\vec{v}^{(n)} \end{bmatrix} = AP$$

and therefore, $D = P^{-1}AP$.

Example

Recall the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

From an earlier problem we know that $\lambda_1 = 10$, $\lambda_2 = \lambda_3 = 1$ and

$$\vec{v}^{\,(1)} = v_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{v}^{\,(2)} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}^{\,(3)} = v_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

From equation **(8)** we have

$$P^{-1}AP = \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 2 & 2 & 1 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 20 & 20 & 10 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 90 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have previously shown that if a matrix A has a full set of eigenvectors, it can be diagonalized by the similarity transformation $P^{-1}AP = D$ (which can also be written in the form $A = PDP^{-1}$). Many problems in mathematics, physics and engineering can be solved by knowing a formula for the n^{th} power of a matrix. One way to establish such a formula is as follows:

$$A = PDP^{-1}$$

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$A^3 = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$A^n = PD^nP^{-1} \qquad \textbf{(9)}$$

If we go back to the previous example,

$$\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}^n = \frac{1}{9}\begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 1^n \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 2\cdot 10^n & -1 & -\frac{1}{2} \\ 2\cdot 10^n & 1 & 0 \\ 10^n & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4(10^n)+5 & 4(10^n)-4 & 2(10^n)-2 \\ 4(10^n)-4 & 4(10^n)+5 & 2(10^n)-2 \\ 2(10^n)-2 & 2(10^n)-2 & 10^n + 8 \end{bmatrix}$$

Homework

1. Show that $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ 1-2^n & 2^{n+1}-3^n & 3^n-2^n \\ 1-2^n & 2^{n+1}-2\cdot 3^n & 2\cdot 3^n-2^n \end{bmatrix}$.

2. Find a formula for the n^{th} power of the matrix $T = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

3. Find a formula for the n^{th} power of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.